DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

291. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

An empty water tank has two inflow pipes A, B, which begin to flow at the same moment. When B, the smaller pipe, has discharged s gallons, and the tank is $1/n$ filled, water from both pipes is turned off. After A, B, have been idle, each as many hours as would suffice it to perform $1/m$ the work done previously by the other pipe, the flow, which is of a uniform rate, is resumed and continued till the tank is filled; B during the second working period has discharged t gallons. (1) What is the capacity of the tank? (2) What would be the capacity if B were an outflow pipe?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let C=capacity, x=number of hours for A to fill tank, y=number of hours for B to fill tank. Then C/x=what A does in one hour, C/y=what B does in one hour. sy/C=time for B to discharge s gallons, $(C-sn)x/(Cn)$ =time for A to discharge $C/n-s$ gallons, sx/Cm=time A is idle, $(C-sn)y/(Cmn)$=time B was idle, $(s+t)y/C$=total time B works, and $(C-s-t)x/C$ =total time A works.

But $\dfrac{(s+t)y}{C}=\dfrac{(C-s-t)x}{C}-\left(\dfrac{(C-sn)y}{Cmn}-\dfrac{sx}{Cm}\right)$.

$\therefore \left(\dfrac{s+t}{C}+\dfrac{C-sn}{Cmn}\right)y=\left(\dfrac{C-s-t}{C}+\dfrac{s}{Cm}\right)x \ldots (1)$.

Also $\dfrac{sy}{C}=\dfrac{(C-sn)x}{Cn}$, or $y=\dfrac{(C-sn)x}{sn} \ldots (2)$.

(2) in (1) gives $C=mn(sn-s-t)+2sn$.

II. $\dfrac{(C+sn)x}{Cn}$=time A works before turned off.

$\dfrac{(C+s+t)x}{C}$=total time A works, $\dfrac{(C+sn)y}{Cmn}$=time B was idle.

$$\therefore \left(\frac{s+t}{C}+\frac{C+sn}{Cmn}\right)y=\left(\frac{C+s+t}{C}+\frac{s}{Cm}\right)x\ldots(3);\ y=\frac{(C+sn)x}{sn}\ldots(4).$$

(4) in (3) gives $C=mn(sn-s-t)-2sn$.

292. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Find the sum of the series $1^2+5^2+14^2+30^2+\ldots+[\frac{1}{6}n(n+1)(2n+1)]^2$.

Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

The differences and the terms of this special series may be arranged as follows for the first seven terms:

$u_1=1^2$		5^2		14^2		30^2		55^2		91^2		140^2	
$u_1=1$		25		196		900		3025		8281		19600	
$\Delta^1u_1=$.	24		171		704		2125		5256		11319		.
$\Delta^2u_1=$.	.	147		533		1421		3131		6063	.	.	
$\Delta^3u_1=$.	.	.	386		888		1710		2932	.	.	.	
$\Delta^4u_1=$.	.	.	.	502		822		1222	.	.	.	.	
$\Delta^5u_1=$.	.	.	.	.	320		400	.	.	.	.	.	
$\Delta^6u_1=$.	.	.	.	.	.	80	.	.	.	.	.	.	

Compute the series for ten terms, or more, and it will be found that Δ^6u_1 are all 80, or constant, therefore all the higher differences vanish. To sum the series we have the value of the leading term and the six leading differences. I have given a general formula for S_n, on page 163, of THE AMERICAN MATHEMATICAL MONTHLY for August-September, 1906, see equation (E). We have:

$$S_n=nu_1+\frac{n(n-1)}{2}\Delta^1u_1+\frac{n(n-1)(n-2)}{3!}\Delta^2u_1+\ldots$$

$$+\frac{n(n-1)\ldots(n-6)}{7!}n^6u_1\ldots(1).$$

From the problem and the above table we have: $u_1=1$, $\Delta'=24$, $\Delta^2=147$, $\Delta^3=386$, $\Delta^4=502$, $\Delta^5=320$, and $\Delta^6=80$. Substitute numerical values in (1), expand the terms, consolidate like terms, reduce, and we have:

$$S_n=\frac{20n^7+140n^6+371n^5+455n^4+245n^3+35n^2-6n}{1260}\ldots(2),$$

$$=\tfrac{1}{1260}[n(n+1)(n+2)(2n+1)(2n+3)(5n^2+10n-1)].$$